# SPECTRAL THEOREM

Let $\Omega \subseteq \mathbb{R}^n$ a set, $\Sigma_\Omega$ $\sigma$-algebra of subsets of $\Omega$

**Def** A projection-valued measure (PVM) is a map

$$E : \Sigma_\Omega \longrightarrow \mathcal{L}(H) \quad \text{s.t.}$$

(i) $E(M)$ is orthog. projection $\forall\, M \in \Sigma_\Omega$

(ii) $E(\Omega) = \mathbb{1}$ , $E(\emptyset) = 0$

(iii) $E$ is "countably additive" i.e.

$$E\Big(\bigcup_{m=1}^{\infty} \Omega_m\Big) = \sum_{m=1}^{\infty} E(\Omega_m)$$

$\forall\, (\Omega_m)_{m\geq 1} \subseteq \Sigma_\Omega$ with $\Omega_m \cap \Omega_n = \emptyset \;\forall m \neq n$

Rk "=" in the strong topology: $\forall x$: $\sum_{m=1}^{M} E(\Omega_m)\,x \xrightarrow{H} E(\cup \Omega_m)x$

(iv) $E(\Omega_1 \cap \Omega_2) = E(\Omega_1)\,E(\Omega_2) \quad \forall\, \Omega_1, \Omega_2 \in \Sigma_\Omega$

**Lemma** $E : \Sigma_\Omega \longrightarrow \{\text{orth. proj}\}$ is PVM

$$\Updownarrow$$

1) $E(\Omega) = \mathbb{1}$

2) $\Sigma_\Omega \longrightarrow \mathbb{R}$ , $M \longmapsto \langle E(M)x, x\rangle$ is a measure $\forall x \in H$

proof: exercise!

**Notation** We denote by $d\langle E(\cdot)x,x\rangle$ the measure $\Sigma_x \to \mathbb{R}$, $M \mapsto \langle E(M)x,x\rangle$

**Key EXAMPLE**

Let $A = A^*$, $A \in \mathcal{L}(H)$

Define via Borel F.C. $\qquad \mathbb{1}_M(x) := \begin{cases} 1, & x \in M \\ 0 & \text{otherwise} \end{cases}$

$$E^A(M) := \mathbb{1}_M(A)$$

i.e. $\langle E^A(M)x,x\rangle = \displaystyle\int_{\sigma(A)} \mathbb{1}_M(\lambda)\, \underbrace{d\mu^A_{x,y}}_{\text{spectral measure of } A} \quad \forall x$

$J$ compact interval : $J \supseteq \sigma(A)$

**Lemma** $E^A : \Sigma_J \longrightarrow \mathcal{L}(H)$ is a P.V.M

**proof** (i) $E^A(M)$ is an orthogonal projection

(ii) $E^A(J) = \mathbb{1}$

$$\langle E^A(J)x,y\rangle = \int_{\sigma(A)} \mathbb{1}_J(\lambda)\, d\mu^A_{x,y} = \int_{\sigma(A)} d\mu^A_{x,y}$$

$$= \langle x,y\rangle \qquad \forall x,y$$

$E(\emptyset) = 0$ : trivial

(iii) $\overset{A}{E}$ count. additive: We take $(\mathcal{S}_n)_{n\geq 1}$ pairwise disjoints: and want to show that

$$\overset{A}{E}\left(\bigcup_{m=1}^{\infty} \mathcal{S}_m\right) = \lim_{N\to\infty} \sum_{m=1}^{N} \overset{A}{E}(\mathcal{S}_m) \quad \text{in the strong topology}$$

Notice that
$$\Pi_{\bigcup_{m=1}^{\infty} \mathcal{S}_m}(\lambda) = \lim_{N\to\infty} \Pi_{\bigcup_{m=1}^{N} \mathcal{S}_m}(\lambda) \quad \forall \lambda$$
$$\sup_{\lambda\in J} \left| \Pi_{\bigcup_{m=1}^{N} \mathcal{S}_m}(\lambda) \right| \leq 1 \quad \forall N$$

By prop. of Borel funct calc: $\forall x\in H$

$$\hat{\Phi}\left(\Pi_{\bigcup_{m=1}^{N} \mathcal{S}_m}\right)x \xrightarrow{N\to\infty} \hat{\Phi}\left(\Pi_{\bigcup_{m=1}^{\infty} \mathcal{S}_m}\right)x$$

$$\overset{A}{E}\left(\bigcup_{m=1}^{N} \mathcal{S}_m\right)x = \sum_{m=1}^{N} \overset{A}{E}(\mathcal{S}_m)x \qquad E\left(\bigcup_{m=1}^{\infty} \mathcal{S}_m\right)x$$

$$\langle \overset{A}{E}\left(\bigcup^{N} \mathcal{S}_m\right)x, y\rangle = \int \Pi_{\bigcup^{N} \mathcal{S}_m} \, d\mu^{A}_{x,y} = \mu^{A}_{x,y}\left(\bigcup^{N} \mathcal{S}_n\right)$$

$$\left(\begin{array}{c}\mathcal{S}_n \text{ pairwise} \\ \text{disjoints}\end{array}\right) = \sum^{N} \mu^{A}_{x,y}(\mathcal{S}_n) = \langle \sum \overset{A}{E}(\mathcal{S}_n)x, y\rangle$$

(iv)
$$\overset{A}{E}(\mathcal{S}_1)\overset{A}{E}(\mathcal{S}_2) = \hat{\Phi}(\Pi_{\mathcal{S}_1})\,\hat{\Phi}(\Pi_{\mathcal{S}_2})$$
$$= \hat{\Phi}\left(\Pi_{\mathcal{S}_1}\cdot \Pi_{\mathcal{S}_2}\right)$$
$$= \hat{\Phi}\left(\Pi_{\mathcal{S}_1\cap \mathcal{S}_2}\right) = \overset{A}{E}(\mathcal{S}_1\cap\mathcal{S}_2)$$

$\blacksquare$

Remark: (1) $\langle E^A(M)x, x\rangle = \int_{\sigma(A)} \mathbb{1}_M(\lambda)\, d\mu_x^A = \int_M d\mu_x^A$

$$= \mu_x^A(M)$$

$M \to \langle E^A(M)x, x\rangle$ is the spectral measure of $A$!

In particular

$$\langle E^A(M)x, x\rangle = \langle E^A(M)E^A(M)x, x\rangle = \|E^A(M)x\|^2 \geq 0$$

(2) From func. calc.

$$\langle Ax, x\rangle = \int_{\sigma(A)} \lambda \, d\mu_x^A = \int_{\sigma(A)} \lambda \, d\langle E^A(\lambda)x, x\rangle \quad \forall x$$

In weak sense: $A$ "=" $\int_{\sigma(A)} \lambda \, dE(\lambda)$

GOAL: Define the spectral integral $\int \lambda \, dE(\lambda)$ where $E$ is a p.v.M and prove that we can write any op $A$ (bd, selfadjoint) as

$$A = \int \lambda \, dE(\lambda)$$

# Spectral integrals

$(\Omega, \Sigma_\Omega)$ , $E$ arbitrary PVM

↳ set ↳ Borel $\sigma$-algebra

**GOAL:** use the PVM to construct operators via integration:

$$\mathbb{I}(f) := \int_\Omega f(t) \, dE(t)$$

for any reasonable function $f$

Which functions?

$$\mathcal{B}_b(\Omega, \Sigma_\Omega) = \left\{ \text{bounded } \Sigma_\Omega\text{-measurable functions on } \Omega, \ \|f\|_\Omega := \sup_{t\in\Omega} |f(t)| < \infty \right\}$$

$$\mathcal{B}_s(\Omega, \Sigma_\Omega) = \left\{ \text{simple functions of } \mathcal{B}_b(\Omega, \Sigma_\Omega) \right\}$$

$$= \left\{ \sum_{r=1}^n c_r \mathbb{1}_{M_r} \quad \text{for some } c_r \in \mathbb{C}, \ M_r \in \Sigma_\Omega \text{ pairwise disjoint} \right\}$$

If $f \in \mathcal{B}_s$, define

$$\mathbb{I}(f) := \sum_{r=1}^n c_r E(M_r)$$

Next step: for general $f \in \mathcal{B}_b$, define

$$\int_\Omega f(t)\, dE(t) := \lim_{n\to\infty} \Pi(f_n)$$

for a sequence of step functions $(f_n)_n$ approximating $f$

**Lemma** $\|\Pi(f)\|_{\mathcal{L}(H)} \leq \|f\|_\Omega \quad \forall f \in \mathcal{B}_s$

**proof** let $f(t) = \sum_{r=1}^{n} c_r \mathbb{1}_{M_r}(t)$

since the sets $M_1, \dots, M_n \in \Sigma_\Omega$ are pairwise disjoints:

$$E(M_l)E(M_r) = E(M_l \cap M_r) = 0 \quad \forall l \neq r$$

then

$$\|\Pi(f)x\|^2 = \left\| \sum_{r=1}^{n} c_r E(M_r)x \right\|^2 =$$

$$= \left\langle \left(\sum_l \overline{c_l} E(M_l)^*\right)\left(\sum_r c_r E(M_r)\right)x, x\right\rangle$$

$$= \left\langle \left(\sum_l \overline{c_l} E(M_l)\right)\left(\sum_r c_r E(M_r)\right)x, x\right\rangle$$

$$= \sum_{l,r} \overline{c_l} c_r \langle \underbrace{E(M_l)E(M_r)}_{\neq 0 \text{ only for } l=r} x, x\rangle$$

$$= \sum_r |c_r|^2 \langle E(M_r)x, x\rangle$$

$$= \sum_z |c_z|^2 \, \| E(M_z)x\|^2$$

$$\leq \sum_z \|f\|_{\ell}^2 \, \| E(M_z)x\|^2$$

$E(M_z)$ orth. proj.

$$= \|f\|_{\ell}^2 \, \| \sum_z E(M_z)x\|^2$$

$$\leq \|f\|_{\ell}^2 \, \| E(\bigcup_{z\geq 1}^{n} M_z)x\|^2$$

$$\leq \|f\|_{\ell}^2 \, \|x\|^2$$

$$\rightsquigarrow \quad \| \Pi(f)\|_{\mathcal{L}(H)} \leq \|f\|_{\ell}$$

☐

Consequently as $B_s$ is dense in $(B_b, \| \ \|_{\ell})$ $\forall f \in B_b$ we can choose a seq $(f_n)_{n\geq 1} \subseteq B_s$ with $f_n \to f$ in $\| \ \|_{\ell}$

$\rightsquigarrow$ $(f_n)_{n\geq 1}$ is a cauchy seq in $\| \ \|_{\ell}$

$$\rightsquigarrow \quad \| \Pi(f_n) - \Pi(f_m)\| \overset{(*)}{=} \| \Pi(f_n - f_m)\|$$

$$\leq \|f_n - f_m\|_{\ell} \xrightarrow{n,m\to\infty} 0$$

$\rightsquigarrow$ $(\Pi(f_n))_n$ Cauchy seq in $\mathcal{L}(H)$, we put

$$\Pi(f) = \lim_{n\to\infty} \Pi(f_n) \qquad (\text{in } \| \ \|_{\mathcal{L}(H)}$$

EXERCISES: 1) check (#)

2) $\mathbb{I}(f)$ does not depend on the approximating sequence.

NOTATION: $$\mathbb{I}(f) = \int_\Omega f(\lambda)\, dE(\lambda)$$

**Prop** (properties of $\mathbb{I}$)

$$\mathbb{I}: \mathcal{B}_b(\Omega, \Sigma_\Omega) \longrightarrow \mathcal{L}(H)$$
$$f \longmapsto \mathbb{I}(f) = \int_\Omega f(\lambda)\, dE(\lambda)$$

fulfills:

(1) algebraic $*$-homeomorphism:

$$\mathbb{I}(\alpha f + \beta g) = \alpha\, \mathbb{I}(f) + \beta\, \mathbb{I}(g)$$
$$\mathbb{I}(1) = \mathbb{1}$$
$$\mathbb{I}(fg) = \mathbb{I}(f)\, \mathbb{I}(g)$$
$$\mathbb{I}(\overline{f}) = \mathbb{I}(f)^*$$

polarization of the measure $d\langle E(\lambda)x, x\rangle$

(2) $$\langle \mathbb{I}(f)x, y\rangle = \int_\Omega f(\lambda)\, d\langle E(\lambda)x, y\rangle$$

(3) $\| \Pi(f)x \|^2 = \int_\Omega |f(t)|^2 \, d\langle E(t)x, x\rangle$

(4) $\| \Pi(f) \|_{\mathcal{L}(H)} \leq \| f \|_\Omega$

(5) $(f_n)_{n\geq 1} \subseteq \mathcal{B}_b(\Omega, \Sigma_\Omega)$ s.t. $\begin{cases} f_n \to f & \text{pointwise} \\ \|f_n\|_\Omega \leq M & \forall n \end{cases}$

Then $\Pi(f_n)x \longrightarrow \Pi(f)x \quad \forall x \in H$

__proof__ prove the properties for simple functions (easy), we hence to extend the properties to arbitrary functions in $\mathcal{B}_b$.

For example, for simple functions:

•) $\Pi(fg) = \Pi(f)\,\Pi(g)$

$f = \sum_r a_r \mathbb{1}_{M_r}$ , $g = \sum_s b_s \mathbb{1}_{N_s}$

$\rightsquigarrow fg = \sum_{r,s} a_r b_s \mathbb{1}_{M_r} \mathbb{1}_{N_s} = \sum_{r,s} a_r b_s \mathbb{1}_{M_r \cap N_s}$

$\rightsquigarrow \Pi(fg) = \sum_{r,s} a_r b_s E(M_r \cap N_s)$

properties of PVM $= \Big(\sum_r a_r E(M_r)\Big)\Big(\sum_s b_s E(N_s)\Big)$

$= \Pi(f) \qquad \Pi(g)$

•) $\| \Pi(f) x \|^2 = \int |f(\lambda)|^2 \, d\langle E(\lambda) x, x\rangle$ $\quad M \to \langle E(M) x, x\rangle$

$f = \sum_r a_r \, \Pi_{M_r}$, then

$$\| \Pi(f) x\|^2 = \sum |a_r|^2 \, \| E(M_r) x\|^2$$
$$= \sum |a_r|^2 \, \langle E(M_r) x, x\rangle$$
$$= \int |f(\lambda)|^2 \, d\langle E(\lambda) x, x\rangle$$

□

EXERCISE: complete the proof

**Thm** (SPECTRAL THEOREM FOR BOUNDED SELFADJOINT OPERATORS)

$A \in \mathcal{L}(H)$, $A = A^*$. Let $J = [a,b] \subseteq \mathbb{R}$ st $J \supseteq \sigma(A)$

Then $\exists!$ PVM $E^A$ on $(J, \Sigma_J)$ st
↳ Borel σ-algebra of $J$

$$A = \int_J \lambda \, dE^A(\lambda)$$

Moreover $\forall f \in \mathcal{B}_b(J)$ we have

$$f(A) = \int_J f(\lambda) \, dE^A(\lambda)$$

defined via Borel func. calc.

Rem $A$ compact : $A = \sum \lambda_i P_{\lambda_i}$

proof existence use Borel func. calc. to define $E^A(M) := \hat{\Phi}(\mathbb{1}_M)$ is a P.V.M.

Use it to construct $\Pi : \mathcal{B}_b(J, \Sigma_J) \longrightarrow \mathcal{L}(H)$

$$f \longmapsto \Pi(f)$$

IF $\Pi(\lambda) = A$, then together with the prop of proportion, it is a Borel funct calculus for $A$

$$\langle \Pi(\lambda) x, y\rangle \overset{(2)}{=} \int \lambda \, d\langle E^A(\lambda) x, y\rangle = \int \lambda \, d\mu^A_{x,y}$$

the measure coincide !

continuous functional calculus

$$= \langle A x, y\rangle$$

But Borel funct calculus is unique! Hence

$$\Pi(f) = \hat{\Phi}(f) \quad \forall f \in \mathcal{B}_b$$

uniqueness let $F$ be an other P.V.M so that

$$A = \int_J \lambda \, dF(\lambda)$$

Again use $F$ to construct $\Pi : \mathcal{B}_b(J) \longrightarrow \mathcal{L}(H)$

It is a Borel func. calculus $\Rightarrow \Pi(f) = \hat{\Phi}(f) \quad \forall f$

$$\rightsquigarrow E^A(M) = \hat{\Phi}(\mathbb{1}_M) = \Pi(\mathbb{1}_M) = F(M) \quad \forall M \in \Sigma_J$$

$\blacksquare$

# APPLICATIONS OF SPECTRAL THEOREM

## (1) CHARACTERIZATION OF SPECTRUM $\quad A \in \mathcal{L}(H),\ A = A^*$

$\sigma_p = \{ \lambda$ eigenvalues: $\ker(A-\lambda) \neq 0 \}$

$\sigma_c = \{ \lambda : \ \ker(A-\lambda) = 0 \ \ \& \ \ \mathrm{Im}(A-\lambda)$ dense $\}$

$\sigma_r = \{ \quad " \qquad " \qquad \overline{\mathrm{Im}(A-\lambda)} \subset H \} = \emptyset$

**Prop** (i) $\lambda_0 \in \sigma(A) \iff \forall \varepsilon > 0 : \ E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \neq 0$

(ii) $\lambda_0 \in \sigma_p(A) \iff E(\{\lambda_0\}) \neq 0$

$\ker(A - \lambda_0) = \mathrm{Im}\ E(\{\lambda_0\})$

(iii) $\lambda_0 \in \sigma_c(A) \iff \lambda \in \sigma(A) \ \& \ \forall x \in H \ E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)x \xrightarrow{\varepsilon \to 0} 0$

**Recall** $E : (\mathcal{J}, \Sigma_{\mathcal{J}}) \longrightarrow \mathcal{L}(H)$, $\quad \mathcal{J} \supseteq \sigma(A)$

$\Omega \longrightarrow E(\Omega)$ $\quad \perp$ project.

**Rem** $\odot$ $E(M) = 0 \implies \mu_x(M) = \langle E(M)x, x \rangle = 0$

$\rightarrow \mu_x$ is supported in $M^c$

$\odot$ $f \in \mathrm{Im}\ E(M) \iff \mu_f = \langle E(\cdot) f, f \rangle$ supported in $M$

$\forall A$ measurable set

$$\Rightarrow)\quad \mu_f(A) = \langle E(A) f, f\rangle = \langle E(A)E(M) f, f\rangle$$
$$= \langle E(A\cap M) f, f\rangle = \mu_f(A\cap M)$$

$$\Leftarrow)\quad f = E(J) f = \underbrace{E(J\setminus M) f + E(M) f}_{\text{orthogonal decomposition}}$$

$$\| E(J\setminus M) f\|^2 = \langle E(J\setminus M) f, E(J\setminus M) f\rangle$$
$$= \langle E(J\setminus M) f, f\rangle = \mu_f(J\setminus M) = 0$$

$$\Rightarrow \quad f = E(M) f$$

<u>proof of proposition</u>

(i) By Weyl $\quad \lambda_0 \in \sigma(A) \iff \exists (f_n)_{n\geq 1}$ $\left| \begin{array}{l} \|f_n\| = 1 \\ \|(A-\lambda_0) f_n\| \to 0 \end{array}\right.$

$\Rightarrow)$ B.C. $\quad \exists \varepsilon > 0 : \quad E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) = 0$

(in part. $\forall x \in H$, $\mu_x$ is supported outside $(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$)

Take Weyl seq: $$\|(A-\lambda_0) f_n\|^2 = \int_{\sigma(A)} |\lambda - \lambda_0|^2 \, d\langle E(\lambda) f_n, f_n\rangle$$

$$\geq \varepsilon^2 \int_{|\lambda-\lambda_0|\geq\varepsilon} d\langle E(\lambda) f_n, f_n\rangle = \varepsilon^2 \mu_{f_n}(|\lambda-\lambda_0|\geq\varepsilon) = \varepsilon^2 \|f_n\|^2 = \varepsilon^2$$

$\mu_{f_n}$ supported in $|\lambda-\lambda_0|\geq\varepsilon$

$\Leftarrow)$ $\forall n$: $\quad E(\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n}) \neq 0$, so $\exists f_n \in \operatorname{Im} E(\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n})$

$$\|(A-\lambda_0) f_n\|^2 = \int_J |\lambda-\lambda_0|^2 \underbrace{d\langle E(\lambda) f_n, f_n\rangle}_{\text{supported in } (\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n})} \leq \frac{1}{n^2} \underbrace{\int d\langle E(\lambda) f_n, f_n\rangle}_{\|f_n\|^2}$$

(ii) $\Rightarrow$) Take $f$: $Af = \lambda_0 f$

CLAIM: $\forall K$ compact, $K \subset J \setminus \{\lambda_0\}$, we have

$$\mu_f(K) = 0$$

Indeed let $\mathrm{dist}(K, \lambda_0) = \delta_K > 0$, then

$$0 = \|(A-\lambda_0)f\|^2 = \int_{\sigma(A)} |\lambda-\lambda_0|^2 \, d\mu_f \geq \int_K |\lambda-\lambda_0|^2 \, d\mu_f$$
$$\geq \delta_K^2 \, \mu_f(K)$$

$$\rightsquigarrow \quad \mu_f(J \setminus \{\lambda_0\}) \underset{\substack{\uparrow \\ \mu_f \text{ inner regular}}}{=} \sup_{K \subset J \setminus \{\lambda_0\}} \mu_f(K) = 0$$

hence by the orthogonal decomposition

$$f = E(J \setminus \{\lambda_0\}) f + E(\{\lambda_0\}) f$$

we have $\|E(J \setminus \{\lambda_0\}) f\|^2 = \mu_f(J \setminus \{\lambda_0\}) = 0$

$\rightsquigarrow \quad f = E(\{\lambda_0\}) f$

$\Leftarrow$) Take $f \in \mathrm{Im}\, E(\{\lambda_0\})$, $f = E(\{\lambda_0\}) f$

$$\|(A-\lambda_0)f\|^2 = \int |\lambda-\lambda_0|^2 \underbrace{d\langle E(\lambda) f, f\rangle}_{\text{supported in } \{\lambda_0\}} = 0$$

(iii) $\lambda_0 \in \sigma_c(A) \iff \lambda_0 \in \sigma(A)$ & $\lambda_0 \notin \sigma_p(A)$ (no residual spectrum)

$$\iff \forall \varepsilon > 0 : \begin{cases} E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \neq 0 \\ E(\{\lambda_0\}) = 0 \end{cases}$$

$\|E(\lambda_0-\varepsilon, \lambda_0+\varepsilon)x\|^2 = \mu_x(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$

$\|E(\{\lambda_0\})x\|^2 = \mu_x(\{\lambda_0\})$

$$\iff \forall x \lim_{\varepsilon \to 0} E(\lambda_0-\varepsilon, \lambda_0+\varepsilon)x = E(\{\lambda_0\})x = 0$$

$\square$

# 2) STABILITY OF SPECTRUM

Given $T \in \mathcal{L}(H)$, $T = T^*$, assume we know everything about $\sigma(T)$.

Now take $V \in \mathcal{L}(H)$, $V = V^*$ and $T + V$

Q: $\sigma(T+V)$ ?

At this level nothing: $V = -T + S$

What about $V$ is "small" perturbation

$\rightarrow$ $V \in \mathcal{L}(H)$, $\|V\| = \varepsilon \ll \|T\|$

$\rightarrow$ $V$ compact operator, (not small in norm)

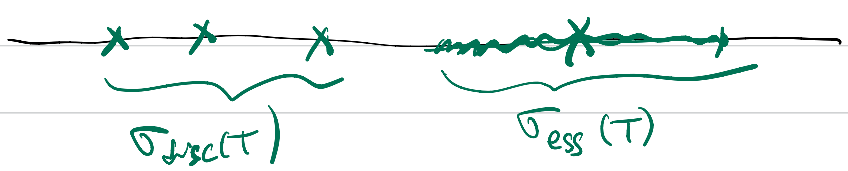
In both cases there are some parts of the spectrum are stable under perturbations.

We need a different decomp. of spectrum

$\sigma_{disc}(T) = \{ \lambda \in \sigma(T) :$ $\lambda$ is isolated eigenvalue of finite multiplicity $\}$

discrete spectrum

$\sigma_{ess}(T) = \sigma(T) \setminus \sigma_{disc}(T) = \{ \lambda \in \sigma(T) ,$ (i) accumulation point (ii) isolated eigenv. with $\infty$ mult. $\}$




$-$ $= \sigma(T)$

$\times$ $=$ eigenvalue ($\ker(A-\lambda) \neq 0$)

$\sigma_{ess}(T)$ is stable under compact perturb.

**Thm** (Weyl's criterion) TFAE (equivalent)

(i) $\lambda_0 \in \sigma_{ess}(T)$

(ii) $\exists (f_n)_{n\geq 1} \subset H$, $\left| \begin{array}{l} \|f_n\| = 1, \quad \|(T-\lambda_0) f_n\| \to 0 \\ f_n \xrightarrow{w} 0 \end{array} \right.$

(iii) $\forall \varepsilon > 0$: $\dim \operatorname{Im} E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) = +\infty$

**proof** (i) $\Rightarrow$ (ii) •) $\lambda_0$ eigenv. of $\infty$ mult $\Rightarrow$ $\ker(T-\lambda_0)$ has $\infty$ dim ✓

•) $\lambda_0$ acc point in $\sigma(T)$, $\leadsto$ take $(\lambda_n)_n \subseteq \sigma(T)$, $\lambda_n \to \lambda_0$

Choose $(\varepsilon_n) \subset \mathbb{R}$, $\varepsilon_n \to 0$ and st $(\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$ are mutually disjoints. Take $f_n \in \operatorname{Im} E(\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$

show that $\left| \begin{array}{l} f_n \rightharpoonup 0 \quad (f_n \perp f_m \quad m \neq n) \\ (T-\lambda_0) f_n \to 0 \quad \text{(similar to previous prop)} \end{array} \right.$

(ii) $\Rightarrow$ (iii) By cont. $\exists \varepsilon > 0$: $\dim \operatorname{Im} E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) < +\infty$

$E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ bd fin. range op $\leadsto$ it is compact

$E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) f_n \to 0$

$$\|(T-\lambda_0) f_n\|^2 = \int_J |\lambda - \lambda_0|^2 \, d\langle E(\lambda) f_n, f_n \rangle \geq$$

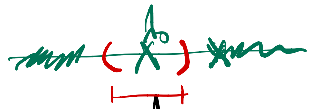
$$\geq \int_{|\lambda - \lambda_0| \geq \varepsilon} \ldots \geq \varepsilon^2 \int_{|\lambda - \lambda_0| \geq \varepsilon} d\langle E(\lambda) f_n, f_n \rangle$$

$$= \varepsilon^2 \int_J - \varepsilon^2 \underbrace{\int_{|\lambda - \lambda_0| < \varepsilon}}_{= \int \mathbb{1}^2_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(\lambda) \, d\langle E(\lambda) f_n, f_n \rangle}$$

$$= \varepsilon^2 \underset{1}{\|f_n\|^2} - \varepsilon^2 \underbrace{\|E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) f_n\|^2}_{\to 0} \quad \overset{\shortparallel}{\|E(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) f_n\|^2}$$

$$\geq \varepsilon^2 / 2 > 0$$

↯




(iii) $\Rightarrow$ (i') B.C. $\lambda_0 \in \sigma_{disc}(T)$, i.e. $\lambda_0$ isolated eigenv. of fin mult.

$$\rightsquigarrow \exists \eta > 0: \quad E(\underbrace{(\lambda_0 - \eta, \lambda_0)}_{\text{no spectrum}}) = E(\underbrace{(\lambda_0, \lambda_0 + \eta)}_{\text{no spectrum}}) = 0$$

$$E((\lambda_0 - \eta, \lambda_0 + \eta)) = \underbrace{E((\lambda_0 - \eta, \lambda_0))}_{=0} + \underbrace{E(\{\lambda_0\})}_{\text{fin dim op}} + \underbrace{E((\lambda_0, \lambda_0 + \eta))}_{=0}$$

$\lightning$

$\square$

We can prove that essential spectrum is stable under compact perturbation.

**Thm (Weyl)** $T \in \mathcal{L}(H)$, $T = T^*$, $V = V^*$ compact.

$$\sigma_{ess}(T) = \sigma_{ess}(T+V)$$

**proof** $\lambda_0 \in \sigma_{ess}(T)$. By Weyl $\exists (f_n)_{n \geq 1}$ Weyl seq, $f_n \rightharpoonup 0$

$$\|(T + V - \lambda_0) f_n\| \leq \underbrace{\|(T - \lambda_0) f_n\|}_{\downarrow \, 0} + \underbrace{\|V f_n\|}_{f_n \rightharpoonup 0,\ V \text{ compact}} \longrightarrow 0$$

$\rightsquigarrow$ $(f_n)_{n \geq 1}$ Weyl seq for $T+V$, $f_n \rightharpoonup 0$

$\rightsquigarrow$ $\lambda_0 \in \sigma_{ess}(T+V)$

$\square$

**Rem 1** We need see $\sigma_{ess}$ & $\sigma_{disc}$

the nature of the spectrum in $\sigma_{ess}(T)$ might change

$\sigma_{ess}(T)$ with no eigenvalues

$\sigma_{ess}(T)$ pure point (dense set of eigenvalues)

EXERCISE: example?

**Rem 2** $\sigma_{disc}(T)$ NOT stable under compact pert
you can create or destroy eigenvalues.

**Ex** $H = L^2(\mathbb{T})$, $T = \cos x$

$$Vu = \delta\left(1-\frac{1}{\delta}\cos x\right)\frac{1}{2\pi}\int_0^{2\pi} u(x)\left(1-\frac{1}{\delta}\cos x\right)dx$$

to

$\sigma(T) = [-1,1]$

$\sigma(T+V)$? $\quad (T+V)[1] = \cos x + \delta\left(1-\frac{1}{\delta}\cos x\right)\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\left(1-\frac{1}{\delta}\cos x\right)dx}_{1}$

$= \cos x + \delta - \cos x = \delta\cdot 1$

$u(x)=1$ eigenv. with $\delta$ eigenvalues

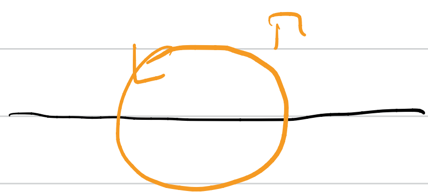
Consider now $V\in\mathcal{L}(H)$, $\|V\|_{\mathcal{L}(H)} = \varepsilon \ll \|T\|$

**EXERCISE**: $\sigma_{ess}$ not stable under bd. small pert.
EXAMPLE?

To say something about $\sigma_{disc}(T+V)$ we need

**Prop** $T\in\mathcal{L}(H)$, $T=T^*$. Let $\Gamma\subseteq\mathbb{C}$ closed path with $\Gamma\subseteq\rho(T)$. Then

$$E(\mathring{\Gamma}\cap\sigma(T)) = -\frac{1}{2\pi i}\oint_\Gamma (T-z)^{-1}\,dz$$




**Rem** $\oint_\Gamma (T-z)^{-1}\,dz = \int (T-\gamma(t))^{-1}\dot{\gamma}(t)\,dt$ with $\gamma(t)$ a parametrization of $\Gamma$ $[0,1]$ the last integral is to be intended as limit in $\mathcal{L}(H)$ of Riemann sums:

$$\lim_{n\to\infty}\sum_j \left(T-\gamma\left(\tfrac{j}{n}\right)\right)^{-1}\dot{\gamma}\left(\tfrac{j}{n}\right)\frac{1}{n}$$

**proof** For $z\in\Gamma$, $z\notin\sigma(T)$. Thus $g_z(\lambda) = \frac{1}{\lambda - z}\in C(\sigma(T))$

$$\Rightarrow \quad (T-z)^{-1} = g_z(T) = \int_J \frac{1}{\lambda - z}\, dE(\lambda) \qquad \text{(sp. theorem)}$$

$$\rightsquigarrow \quad -\frac{1}{2\pi i}\oint_\Gamma (T-z)^{-1}\, dz = -\frac{1}{2\pi i}\oint_\Gamma \left( \int_J \frac{1}{\lambda - z}\, dE(\lambda) \right) dz$$

$$= \int_J \Bigg( \underbrace{+\frac{1}{2\pi i}\oint_\Gamma \frac{1}{z-\lambda}\, dz}_{= \begin{cases} 1 & \text{if } \lambda \in \text{interior of } \Gamma \\ 0 & \text{otherwise} \end{cases}} \Bigg)\, dE(\lambda)$$

$$= \int_J \mathbb{1}(\mathbb{R} \cap \mathring{\Gamma})\, dE(\lambda) = E(\mathring{\Gamma} \cap \sigma(T))$$

↳ we know $E(\Omega) \neq 0$ iff $\Omega \cap \sigma(T) \neq \emptyset$

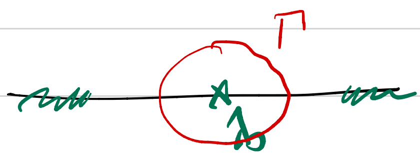
$\square$

In particular if $\sigma(T) = \sigma_1 \cup \sigma_2$, $\sigma_1 \cap \sigma_2 = \emptyset$ and $\Gamma$ encloses $\sigma_1$, we have

$$E(\sigma_1) = -\frac{1}{2\pi i}\oint_\Gamma (T-z)^{-1}\, dz$$

**Thm** $T = T^*$, $T \in \mathcal{L}(H)$, $\lambda_0 \in \sigma_{disc}(T)$ eigenv. of mult 1. Take $V = V^*$, then $\forall \varepsilon$ suff. small, $T + \varepsilon V$ has eigenv. $\lambda_\varepsilon$ close to $\lambda_0$.

**Proof** Take $\Gamma$ isolating $\lambda_0$.




First we prove that for $\varepsilon$ suff small, $\Gamma \subseteq \rho(T + \varepsilon V)$

Use Neumann series : (exercise)

Now construct $P_\varepsilon := -\frac{1}{2\pi i} \oint_\Gamma (T+\varepsilon V - z)^{-1} \, dz$

$P_\varepsilon$ is an orthogonal projection.
We want to prove that $P_\varepsilon \neq 0$ and $\dim \operatorname{Im} P_\varepsilon = 1$
$\dim E(\sigma(T) \text{ inside } \Gamma) = 1 \rightsquigarrow$ there is eigenvalue of $T+\varepsilon V$

$$P_\varepsilon - P_0 = -\frac{1}{2\pi i} \oint_\Gamma \left( (T+\varepsilon V - \gamma)^{-1} - (T-\gamma)^{-1} \right) d\gamma$$

$$= \frac{1}{2\pi i} \oint_\Gamma (T+\varepsilon V - \gamma)^{-1} \varepsilon V (T-\gamma)^{-1} \, d\gamma$$

$\hookrightarrow$ resolvent identities: $(A-\gamma)^{-1} - (B-\gamma)^{-1} =$
$= (A-\gamma)^{-1} (B-A)(B-\gamma)^{-1}$

$$\rightsquigarrow \| P_\varepsilon - P_0 \|_{\mathcal{L}(H)} \leq \oint_\Gamma \frac{1}{\operatorname{dist}(\sigma(T+\varepsilon V), \gamma)} \varepsilon \|V\| \frac{1}{\operatorname{dist}(\sigma(T), \gamma)} \, d\gamma$$

$$\leq \frac{|\Gamma| \, \varepsilon \, \|V\|}{\operatorname{dist}(\sigma(T+\varepsilon V), \Gamma)} \frac{1}{\operatorname{dist}(\sigma(T), \Gamma)} \ll 1$$

$\rightsquigarrow \dim \operatorname{Im} P_\varepsilon = \dim \operatorname{Im} P_0 = 1$
by the following lemma.

**Lemma** $P, Q$ projections. If $\|P - Q\| < 1$
$\Rightarrow \dim \operatorname{Im} P = \dim \operatorname{Im} Q$

*proof* B.C. $\dim \operatorname{Im} P > \dim \operatorname{Im} Q$. Then $\exists\, u \neq 0$
with $u = Pu$, $Qu = 0$.
Then $\|(P-Q)u\| = \|u\|$ $\lightning$